Euler and the Konigsberg Bridges: Some Lessons for the Philosophy of Mathematics

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Abstract

Platonism has exercised a reign over the philosophy of mathematics ever since the times of classical Greece. Whilst this dynasty has seldom been exercised either in a 'pure' form or without rivals it continues to exercise a powerful influence both amongst philosophers and mathematicians today. In particular the legacy of Platonism, in combination with the paradigmatic idealisation of Euclidean Geometry developed within the Cartesian and subsequent rationalism of the seventeenth and eighteenth centuries, culminated in the philosophical position of Immanuel Kant. This attempt to maintain an aprioristic rational character to Pure Mathematics came to grief with the mathematical developments of non-Euclidean Geometries. However, the goal of building an infallible mathematical edifice upon sure, sound and logical foundations has continued to dominate the philosophy of mathematics for the past century.

For much the greater part, however, this philosophical trend has had little following amongst mathematicians. In this respect, taking its cue from the revolution in the philosophy of science of some thirty years ago, we are witnessing a new twist to overturn the long reign of Platonism. A number of authors have proposed that the philosophy of mathematics, too, should take a serious and hard look at what mathematicians actually do. In a recent book, ''What is Mathematics, Really?" one of these authors – Reuben Hersh – proposes such a standpoint – one that he labels 'humanism'. This paper, in many ways, sets out to be a critically appreciative review of this book. On the one hand it applauds the thesis to the effect that mathematics is an integral part of human culture that is fallible and has an ongoing history. On the other hand it seeks to offer a corrective to the other main thesis that 'mathematics is a social-historic reality with no hidden meaning or definition beyond this social-historic-cultural context'.

The chief critical tool in this latter respect involves a case-study of a seminal paper in the development of modern mathematics. In 1735, the Swiss Mathematician Leonhard Euler delivered a paper entitled 'The Seven Bridges of Konigsberg', in which he showed that the well-known pastime of citizens attempting to find a path crossing all the Konigsberg bridges only once, was impossible. More significantly for the history of Mathematics, the content and methods developed in the paper were pioneering with respect to the development of both modern topology and graph theory.

Arising from this case study the philosophical issues addressed include the relationship between "Pure" and "Applied" Mathematics, the ontological reality/ideality of mathematical entities and the abstract/empirical nature of mathematical knowledge. The course of this discussion involves a critical evaluation of Sir Karl Popper's "World 3" hypothesis and an attempt to develop a view of abstraction that is more in tune with Aristotle than Plato.

1. A Look at the Current Situation of the Philosophy of Mathematics.

Until very recently the Philosophy of Mathematics has been dominated by Foundationalism. Logicism, Formalism and Intuitionism, each in their different ways, have sought to preserve the spirit of epistemological certainty attributed to the kind of sophisticated mathematical system exemplified by Euclidean Geometry. However, the long- held view that Euclidian Geometry provided the paradigm of rationalist epistemology underwent a major setback with the discoveries of the possibilities of non-Euclidean Geometries. The significance of this development was compounded by the success of one of these geometries - Riemannian Geometry - providing the geometrical basis for the physical space of Einstein's General Theory of Relativity.

Thus, as Euclidean Geometry was unable to provide the kind of certain epistemological foundation for Kantian epistemology, alternative firm foundations were sought in the form of Logicism, Formalism and Intuitionism. The basis for the Logicist programme was a revamped, general and rigorous attempt to develop Mathematics from Logic – invoking the general idea of a ***set*** as its basic notion. The basis for the Formalist program was to view mathematical systems as humanly constructed formal languages capable of varying degrees of complexity and which were considered to exemplify the intuitive notion of formal proof. The intuitionist program, on the other hand was deeply suspicious of utilising the logical principle of the law of the excluded middle in respect of sets of entities of infinite order. In their endeavour to keep Mathematics unpolluted by the liberal use of proof techniques heavily dependent upon such arguments, they sought to preserve the Kantian heritage of 'the intuition of the human activity of counting' as providing the certitudinal basis for mathematics together with a rigorous attempt to humanly construct the rest of the human mathematical architecture from this intuition in a way that avoided the wanton use of the logical law of the excluded middle in respect of infinite sets.

Each of these research programs have hit major brickwalls over the past hundred years or so: the contradictions within Russell's Set Theory for Logicism and the significance of Godel's Incompleteness Theorems for Formalism were comparable to the impact of the Pythagorean discovery that the square root of the number 2 is not rational. However, despite these setbacks, the philosophy of mathematics continues to be dominated by "Neo-Fregeanism" - a form of Platonism in conjunction with Foundationism that pays little attention to what mathematics – outside of set theory – is all about.

It is against this background that we need to understand the recent contribution of people like Imre Lakatos, Philip Kitcher and Reuben Hersh. Hersh has a background of a mathematician within the field of Differential Equations. He developed interests in the philosophy and history of mathematics and embarked upon a program of trying to teach the foundations and philosophy of mathematics. His first two books – *The Mathematical Experience* and *Descarte's Dream,* jointly authored by P.J. Davis, are attempts to convey something of the truly creative character of living mathematics as it is practised by mathematicians pursuing their interest in solving problems and at the same time attempting to relate their insights to the full realities of the broader human condition. In his more recent book – *What is Mathematics Really?* – he makes a major effort to attack Foundationism as well as its roots in Platonism.

He paints with a very broad brush on a very large canvas – the whole history of mathematics and philosophy. He adopts the term 'foundationism' from Lakatos. As such it refers to the work of Gottlob Frege in his prime, to that of Russell at the height of his logicist phase, to Brouwer's intuitionism as well as to Hilbert's formalism. Lakatos saw that despite their disagreements, they were agreed on one central thing*: Mathematics must have a firm foundation.* Their differences arose from the issue as to what that foundation should be. Hence the term *foundationism* to describe the view that the logical/formal language structure of Mathematics needs a solid firm foundation to guarantee its infallibility. His attack on foundationism has three prongs:

- Each of the three programs for developing this kind of foundation has come to grief upon logical contradictions that have undermined the whole venture.
- In general the programs, whilst in some cases having a profound influence upon the teaching and presentation of mathematics, have generally had little to do with the actual practice of mathematics. Moreover, for the greater part, the influence that foundationism has had upon the teaching and practice of mathematics has not been in the best interests of fostering the well-being of the subject.
- Foundationism has ancient roots. Frege, Hilbert and Brouwer all depend upon Kant. Kant stands in a tradition that attempts to salvage an apriori rationalism for mathematics from the rationalistic metaphysics of Descartes, Spinoza and Leibniz. These, in their turn, depend upon Aquinas, Augustine, Plato and Pythagoras.

In this light, Hersh attempts to give an historical survey of the way in which the Platonic-Pythagorean tradition has shaped *the mainstream* philosophy of mathematics up to the present day. Central to his general thesis in this respect is his contention as to the way in which religion and theology have played a central role in this 'mainstream'.

> *We will find that the roots of foundationism are tangled up with religion and theology. In Plato and Pythagoras, this intimacy is public. In Kant, it's half covered. In Frege it's out of sight. Then in Georg Cantor, Bertrand Russell, David Hilbert, and Luitjens Brouwer, it pops up like a jack-in-the-box.(p92)*

As an illustration of the continuing quest for 'religious' certainty related to the philosophy of mathematics associated with the mainstream in the age of religious skepticism, he quotes from Russell as follows:

> *I wanted certainty in the kind of way in which people want religious faith. I thought that certainty is more likely to be found in mathematics than elsewhere. But I discovered that many mathematical demonstrations, which my teachers expected me to accept, were full of fallacies, and that, if certainty were indeed discoverable in mathematics it would have to be in a new field of mathematics, with more solid foundations than those hitherto thought secure.* (Bertrand Russell, 'Reflections on My Eightieth Birthday' in *Portraits from Memory)*

The intimate connection between mathematics, religion and mysticism begins with Pythagoras and is continued in Plato. In this respect the paradigm of Plato's Ideas was mathematical. Following the Pythagoreans, with whose philosophy he seems to have been particularly enchanted, Plato understood the universe to be organized in accordance with the mathematical ideas of number and geometry. These Ideas are invisible, apprehensible by intelligence only, and yet can be discovered to be the formative causes and regulators of all empirical visible objects and processes. At the same time, however, circles, triangles and numbers were numinous and transcendent entities that had a divine quality existing independently both of the human mind and of the phenomena perceived by it. It was for this reason, for example, that the only possible paths that could be followed by the planets within the perfect heavenly realm were deemed to be circles.

This Platonism entered the Judaeo-Christian tradition through a range of Patristic theologians, the most influential within the Western Church being Augustine. Within his strain of thought, the Platonic Ideas became incorporated into the 'mind of God'. This implied that through the exercise of human reason one could access the principles by which God ordered the universe. For this purpose, no experiments, no testing of theories was required. It could be known *apriori*. This was deemed to be true particularly of mathematics and theology, with an intimate tie being established between the two. Whilst *this apriorism* was challenged in some respects by the late mediaeval influence of the more empirical Aristotelean mode of thought, its influence was only blunted, not curtailed. However, mathematics was never the key to unlocking the secrets of the order of the universe with Aristotle and it was a further influence of Renaissance Platonism along with that of the mathematical physics of Archimedes that was instrumental in the emergence of the significant role of mathematics in the development of modern science. In this respect, although Descartes rejected the divine and independent existence of Plato's Ideas, he still sought to establish human knowledge upon a sure and sound rationalistic basis that gave mathematics a sure and firm foundation. This paved the way for Kant and the modern foundationists.

This background of Platonism together with Foundationism is labelled by Hersh as 'the mainstream'. Opposing the mainstream Hersh cites a range of authors from Aristotle to the present day. He describes them as 'humanists and mavericks', and the principle criteria for this qualification is 'the idea of mathematics as a human creation'. The major champion of the humanistic philosophy of mathematics he wishes to espouse is Imre Lakatos, of whom he writes:

> *If we wanted to separate the history of humanistic philosophy of mathematics from its prehistory, we could say that Aristotle, Locke, Hume, Mill, Peirce, and Wittgenstein are our prehistory. Our history starts not with Frege, but with Lakatos. (p208)*

In particular, he refers to the remarks of Lakatos in the introduction to *Proofs and Refutations,* in which the latter gives 'a blistering attack' on Formalism – that school which, according to Lakatos, 'tends to identify mathematics with its formal axiomatic abstraction and the philosophy of mathematics with metamathematics'. As such 'formalism disconnects the history of mathematics from the philosophy of mathematics. Formalism denies the status of mathematics to most of what has been commonly understood to be mathematics, and can say nothing about its growth'.

However, Hersh believes that Lakatos did not go far enough. Moreover he regrets the latter's excursion into the Philosophy of Science, a fact that, in view of his untimely death, meant that he was unable to complete the revolution in the philosophy of mathematics that began and, according to Hersh, almost ended with *Proofs and Refutations*.

In this respect Hersh advocates a similar reversal in the Philosophy of Mathematics that occurred in the Philosophy of Science some twenty years ago. In broad terms, the Logical Positivism of the latter parallels the Formalism of the former, and Lakatos plays the intermediary role in respect of the Formalists and Hersh's 'humanistic philosophy' to to that played by Popper between the Logical Positivists and the likes of Kuhn and Feyerabend.

A summary of his viewpoint, which I shall set out as Thesis One, is as follows:

Thesis One

1 *Point 1 is that mathematics is a social-historic reality. This is not controversial. All that Platonists, formalists, intuitionists and others can say against it is that it's irrelevant to their concept of philosophy.*

2 *Point 2* **is** *controversial: There's no need to look for a hidden meaning or definition of mathematics beyond its social-historic meaning. Social-historic is all its need to be. Forget foundations, forget immaterial, inhuman 'reality'.*

There is much in Hersh's work that is very commendable. His work is particularly significant to mathematicians interested in the philosophy of mathematics. In this respect Point 1 is very relevant to an appreciation of the character of mathematics. The latter **is** a human cultural activity and any philosophy of mathematics that does not see this as a fundamental point is to be found wanting. In the comparative neglect of the actual history of mathematics on the part of Foundationism, Hersh's critique is thoroughly justified.

At the same time it seems to me that his Point 2 is downright false. In saying this I do not wish to try to continue the fallacies of foundationism in its vain attempt to salvage epistemological certainty. My concern is that mathematics **is** distinguishable from other branches of human cultural activity, and one of the ongoing tasks of philosophy is to articulate this demarcation with the clarity and rigour to be expected of an academic discipline with its long heritage.

One of the annoying things about Hersh's bold conjectures is that he does not discuss or develop his idea of 'cultural – socio –historic' in any detail. In an endeavour to bring some clarity to this matter I shall begin with an attempt to develop this idea from a critical consideration of Popper's "World 3" hypothesis. Because of its emphasis upon theories as an objective contribution to the world of human cultural products, this hypothesis, as developed by Sir Karl in two of his essays in *Objective Knowledge,* has a clear bearing upon this topic. Moreover, because the hypothesis purports to be a reform of the Platonic world of abstract entities, it has an added relevance to the present discussion.

2. The World of Cultural Objects and Popper's 'World 3' Hypothesis.

Mathematical theories are cultural objects. Does this warrant them being considered inhabitants of Popper's 'World 3'? I shall introduce the notion of 'World 3' in the words of Popper himself:

> *To explain this expression I will point out that, without taking the words 'world' or 'universe' too seriously, we may distinguish the following three worlds or universes: first, the world of physical objects or of physical states; secondly, the world of states of consciousness, or of mental states, or perhaps of behavioural dispositions to act; and thirdly, the world of objective contents of thought, especially of scientific and poetic thoughts and of works of art.*
>
> *It is not part of my view or my argument that we might not enumerate in different ways, or not enumerate them at all. We might, especially, distinguish more than three worlds. My term 'the third world' is merely a matter of convenience.*
> ('Epistemology Without a Knowing Subject', *Objective Knowledge*, pp 106-107)

Popper's principal interest in developing this theory is his realist, fallible, critical view of the growth of knowledge. Whilst I take his point to the effect that the central issue is not that of a dispute about words, my major concern here is ontological rather than linguistic or epistemological.

In this respect, let me begin by pointing out an important ambiguity at the very outset of Popper's discussion. 'World 3' is identified as '*objective contents of thought*, especially of scientific and poetic thoughts and works of art'. (Emphasis his *Ibid* p106)

To claim that works of art are contents of thought is not only mistaken, it is also illustrative of the kind of problem that Popper wants to overcome – the subjectivism or relativism of the artistic merits of works of art.

The mode of existence of a painting, a play, a symphony, a dance or a sculpture is not primarily in the form of ideas or contents of thought. Their primarily aesthetic character cannot be separated from their physical mode of existence – in other words the 'World 1' character of these works of art is more significant than any 'contents of thought'.

Thus, if 'a work of art' is to be a candidate for an inmate to 'World 3', then the definition of the latter has to be more general than 'the world of the contents of thought'. It seems to me that Popper himself made some good suggestions in this direction with his reference to biological analogies in respect of human cultural products. Thus he writes:

> *A biologist may be interested in the behaviour of animals: but he may also be interested in some of the non-living structures which animals produce, such as spider's webs, or nests built by wasps or ants, the burrows of badgers, dams constructed by beavers, or paths made by animals in forests. (*Ibid p112*)*

Again, these products of animal formation belong to 'World 1', in much the same way that paintings and symphonies do. He goes on to say that

> *These simple considerations may of course be applied to products of human activity, such as houses, or tools, and also to works of art. Especially important for us, they apply to what we call 'language', and to what we call 'science' .*
> *(*Ibid p113*)*

He again seems to suggest that the former – houses, tools as well as works of art – belong to 'World 3'. Clearly none of them have a primary mode of existence that can be said to be in the form of 'contents of thought' as opposed to the 'physical' mode of 'World 1'. Upon the basis of these observations, it would seem that the 'physical' character of 'World 1' is also in need of some critique. Unfortunately, as formulated by Popper, it would still seem to carry the weight of the seventeenth/eighteenth century distinction between *primary* (those belonging to the thing itself) and *secondary* (those attributed to the thing by the human mind) qualities. Clearly a symphony, a painting, a sculpture, a bird's nest or a house all have physical properties. However, the former three would appear to gain their objective character from the way the aesthetic properties coalesce with the physical and sensitive properties of the sounds, colours and shapes of things. A better description of 'World 1' might therefore be 'concrete reality' rather than physical reality. The description 'concrete' simply refers to the variety of things of our ordinary experience – things that do indeed have a physical existence, but whose properties, at least *prima facie,* embrace biological, aesthetic and other kinds of properties that may not automatically be said to be reducible to the physical properties in the manner of the primary/secondary qualities distinction.

In this sense we may refer to things such as tables, tools, houses, computers and works of art as **cultural objects** belonging to 'World 1' redefined as **the concrete world.** These objects, whilst having physical properties, are yet recognisably different from unhewn stones, trees and naturally occurring metal ores. This difference arises from the concrete effects of the exercise of human formative power upon what might be said to be 'natural' things /events/processes. Moreover, the exercise of this formative power arises from the imaginings, the vision, the plans and the intentions of human beings as they take place within 'World 2' – a world which, in contrast to 'World 1', is not directly accessible to the senses.

Popper's chief concern in proposing 'World 3' is that of emphasising that human knowledge is not simply a the matter of subjective states of the human person in respect of 'World 2'. Thus Popper writes to the effect that:

> *My first thesis involves the existence of two different senses of knowledge or of thought: (1) knowledge or thought in the subjective sense, consisting of a state of mind or of consciousness or a disposition to behave or to react, and (2) knowledge or thought in an objective sense, consisting of problems, theories, and arguments as such. (pp109-110)*

Now I believe that Popper's claim, in this respect, to be both important and correct. However, I also believe that his attempt to introduce a 'World 3' differing from 'World 1' and 'World 2' raises many more problems than it solves. Such cultural objects as houses, tools, books and works of art are readily located in 'World 1'. The contents of thought as well as the content of theories are also readily located in this world through oral transmission and written records. The question as to **just where** 'the contents of thought' may be located in any culturally objective sense – apart from their objectivisation in language – raises the important issue of Platonism. In this respect it is significant that Popper does relate his view of 'World 3' to Plato's world of ideas, with the principal difference to be found in the thesis that 'my third world is man-made and changing' whereas 'Plato's third world was divine'. (pp122-123)

Popper's good intentions of seeking a cultural objectivity to scientific and other contents of thought within the realm of cultural objects leads to the following dilemma:

> The need to remove the ambiguity in Popper's view of 'World 3' so that it is **not** identified with *the contents of thought* leads in the direction of viewing his 'World 3' as a 'world of cultural objects' that is located within 'World1' but characterised by a particular type of property of the things /events/processes in 'World 1' – the property of human formative power. In this respect 'World 1', redefined as 'the world of concrete things /events/processes complete with their diverse range of properties' so that it includes all the products/objects of human cultural formative activity, including scientific theories, does away with the need for a 'World 3' distinct from 'Worlds 1&2' , but at the same time requires a redefinition of 'World1'
>
> On the other hand, maintaining Popper's own emphasis upon 'World 3' as typified by the contents of thought and of scientific theories in particular, leads in the direction of a world of ideas that, whilst created by the 'World 2' activities and purported to be about 'World 1' entities, is yet somehow disembodied from both 'Worlds 1 and 2'. This suggests a notion of an abstract world of ideas or entities whose meaning requires explication. On

> the face of it, these ideas are both created and accessed by human beings through an exercise of reason, but with claims to having a distinct form of existence from that of Worlds 1&2. In other words, albeit humanly created, 'World 3' still has all the problems of Platonism.

Of course, it may be claimed that all that is implied by this is that 'World 3' and 'World 1' are not mutually exclusive. As a result of the intentions/plans/habits/skills/knowledge of 'World 2' activity, items within 'World 1' take upon the character of an object in 'World 3' - a loose collection of stones in 'World 1 can become a house or fence – an item in 'World 3'. A block of marble – another item in 'World 1' may be carved and chipped to yield a sculpture – a 'World 3' object. In the process of transforming 'World 1' objects into 'World 3' objects, of course, much of the knowledge and skill necessary for the appropriate 'World 2' activity to accomplish these goals, is obtained from a process of learning involving the utilisation of such 'World 3' objects as books, libraries, teachers and training schemes.

The point then is: is there any sense in which a 'World 3' object is not also a 'World 1' object? The 'social-cultural' nature of the knowledge recorded and assembled in books and libraries' is Popper's chief point in proposing the hypothesis, and 'the world of theories' – as the objective content of thought comprising this knowledge – is his prime concern. To claim that 'the contents of thought' have an objective reality apart from the world of cultural objects as concrete things (books, journals etc) within 'World 1' and the 'subjective activity' of people in 'World 2' does indeed raise the question as to their ontological status, with the only possible candidate seeming to be some form of abstract entity that has an ontological home in a world of ideas that has a distinct existence from their instantiation in the things/processes/events of our experience. In this particular respect, it would be quite possible to view Plato's world of abstract ideas as a world constructed by humans via a process of abstraction. In other words, Popper's claim to the effect that the chief difference between his 'World 3' as *man-made and changing* and Plato's as *divine* still leaves open the question of the existence of world of abstract entities that is distinct from the concrete world of our experience.

In an effort to deal consistently with the issue of Platonism and abstract entities, I propose that 'the world of cultural objects' – including the objectivisation of contents of thought an well as scientific theories - be viewed as properties of concrete things/events/processes of 'World 1', with this class of properties being defined by the evidence of the exercise of human formative power. It is through the exercise of such power that a block of marble becomes a sculpture, a pile of stones becomes a fence or a house. It is also by the exercise of such formative power that a theory such as Newton's becomes objectified in a book like *Principia Mathematica,* or books are collected and assembled in a library. In this way books and libraries are enabled to play the kind of role required by Popper in his concern for objective (and fallible) knowledge. However, rather than introduce a separate 'World 3', we simply revise our view of 'World 1' so that it makes a clean break with the legacy of the distinction between primary and secondary qualities.

A significant part of Hersh's proposal for a 'humanist' philosophy of mathematics is that it include a focus upon the *social reality* of the way in which living mathematics operates within the human community: - both a 'back-room community' of working mathematicians and a 'front-room community' in which the former interacts with the rest of human life in the broader community of education, science and technology (pp35-47). That this is an important fact of the reality of human mathematical life cannot be denied, and, as such, I think Hersh is quite correct in insisting that it be brought into the scope of philosophical scrutiny. I would simply wish to point out that the social realities – the interactions between people holding various offices that give them certain powers – are all accessible to human experience and observation, and, as such also part of 'World 1' understood as the concrete world of our everyday experience. The fact that a certain level of human knowledge is generally required to be able to appreciate the details of what is taking place in such interactions does not alter this claim. For example, as Hersh points out:

> *At any meeting of the American Mathematical Society, any contributed talk is understandable to only a fraction of those present.(p38)*

Whilst this claim may be true of the content of the talk, few would not be able to recognise the appropriate/inappropriate exercises of power by the chairperson, speaker and responses. To appreciate both the content of the talk as well as the realities of such social interactions requires the kind of knowledge needing to be acquired by means of books, courses, libraries, parents or teachers - Popper's 'World 3'. However, the point here is that the relevant things/events/processes all take place within 'World 1'. To appreciate the social interactions taking place within such events simply requires us to focus upon the appropriate kind of property (one that differs from the physical or aesthetic) within the things/ events/processes of the concrete world of our experience.

Accordingly, with particular reference to **Thesis One** - of Reuben Hersh - above, I propose that we consider the *socio-cultural-historical* realm within which to locate the (fallible) mathematical activity mentioned by him, in accord with the following theses:

Thesis Two

1 *'World 1' is the world of concrete reality - experienced by us as the things/ events/processes having a coherence that involves a diversity of properties. As such a prima facie case can be made for the view that these things/events/processes of concrete reality are 'anchored in' physical properties, but are not exhausted by them. In particular the world of 'cultural objects and social interactions' is part of this concrete world of our experience.*

2 *'World 2' is the world of human mental states which are significant in their interaction with 'World 1' as cultural subjects to produce the world of cultural objects within the context of social interactions.*

3 *The world of cultural objects includes Popper's 'world of the contents of thought' as well as houses, sculptures, paintings and many other things/events/processes within 'World 1' having the properties of cultural formation. In this respect the 'objective content of thought' emphasis given by Popper needs some modification. The objective character of theories or' contents of thought' do not exist in a separate 'World 3' , but in the form of such objectified lingual/symbolical cultural products such as oral speech, books and libraries that are accessible by 'World 2' activities. As such they belong to the concrete 'World 1' of things/events/processes having a diversity of properties. As such this collection of things/events – the world of cultural objects that have been shaped by human beings - plays a significant role in the ongoing development of human culture.*

4 *The contents of the thought of scientific/mathematical/philosophic theories arise partly from the experience of the relevant properties/entities of 'World 1' and partly from the ongoing 'semi-independent' character of the theories as cultural objects once they have been developed. The development of mathematical theories, as it involves the precise roles of these*

two features of 'World 1' as well as certain kinds of activity in 'World 2', is the chief concern of this paper.

With this broad conception of the 'social-cultural-historic reality' we shall seek to look at a particular case study so as 'to learn from what mathematicians actually do' in an attempt both to gain some insight as to the character of mathematics as set out in Thesis Two. At the same time, bearing in mind the radical claim of Hersh in Point 2 of Thesis One, we shall seek to use this study to help clarify some aspects of the relationship between the world of cultural objects and other the kinds of properties of concrete entities comprising 'World 1'.

3. Learning from the Practice of Mathematics.

Taking its cue from the need to draw lessons from the actual history and practice of mathematics, we shall take a serious look at Euler's work in solving the Konigsberg Bridges problem. To further motivate it however, we shall take a look at a modern practical problem.

3.1. A Modern Practical Problem – Picking up the Garbage.

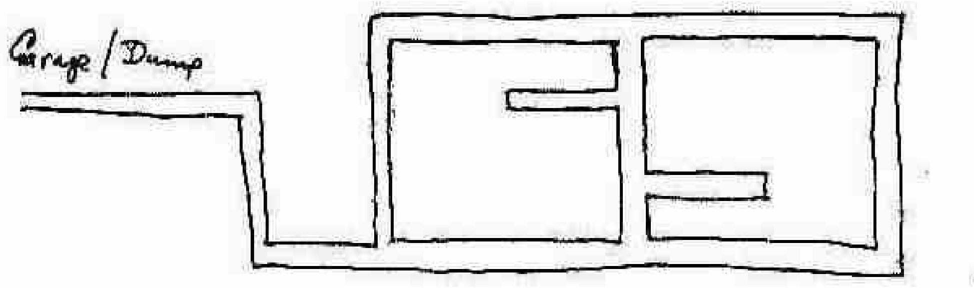
Consider a contemporary practical problem that is capable of mathematical analysis via Operational Research. It concerns the collection, transport and disposal of garbage by means of trucks that pick up the garbage left out on the street by residents on an appropriate day of the week. For simplicity we shall assume that it is collected in large bins supplied by the City Council. These bins - full with 'goodies' - are slotted into an automatic lifter and emptied into the body of the truck, implying that it is necessary to go along one side of a street at a time.

The Operational Research problem is as follows:

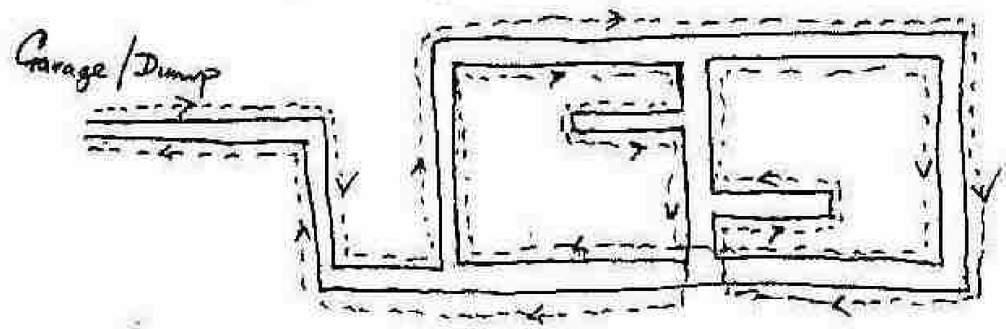
- How should the city-scape best be divided up into zones to have the garbage collected by a single truck on a given run to and from the depot, including a visit to the dump. This issue depends upon the average volume of garbage per household, the volume of the collection trucks as well as the positioning of the garbage disposal dump and garaging facilities for the trucks. It should also depend upon a design of the routes that would at least minimise and at best exclude the necessity of garbage trucks doubling back upon them selves, covering large sections of street-scapes without actually picking up garbage.
- Given these objectives and constraints how should the routes taken by the trucks be designed so that the truck collects all the garbage with a minimum expenditure of fuel, distance and time?

I will give a simple illustration of the way this problem might be tackled:

Consider the following small street-scape:



It is evident that our problem might be solved as follows:



Now let us consider the situation in which one side of a long street comprises a park: there are no residents and no garbage bins to be picked up on one side of this section of road. Is it now possible to design a route for our garbage truck that does not require us to travel along the relevant section of street twice – once without picking up any garbage from the borders of the park?

We could try to answer this question by attempting to experiment and find a route satisfying the conditions. On the other hand we could model the problem as a Network and utilise some of the results of theorems concerning Euler circuits within Graph Theory:

Theorem 1:

> A continuous path connecting all the nodes of a graph, beginning and ending at the same node, utilising each and every arc only once exists only if the order of all the nodes is even.

Theorem 2:

> A continuous path connecting all the nodes of a graph, allowing for the possibility of beginning and ending at different nodes, utilising each and every arc only once, exists if the number of nodes with odd orders is either zero or two. In the latter case the continuous path must begin at one odd node and end at the other.

If the trucks are garaged at the garbage dump, then a solution using Theorem 1 would be desirable. On the other hand, if the trucks were garaged at the a central City Council location, well away from the dump, then a solution utilising Theorem 2 would be desirable.

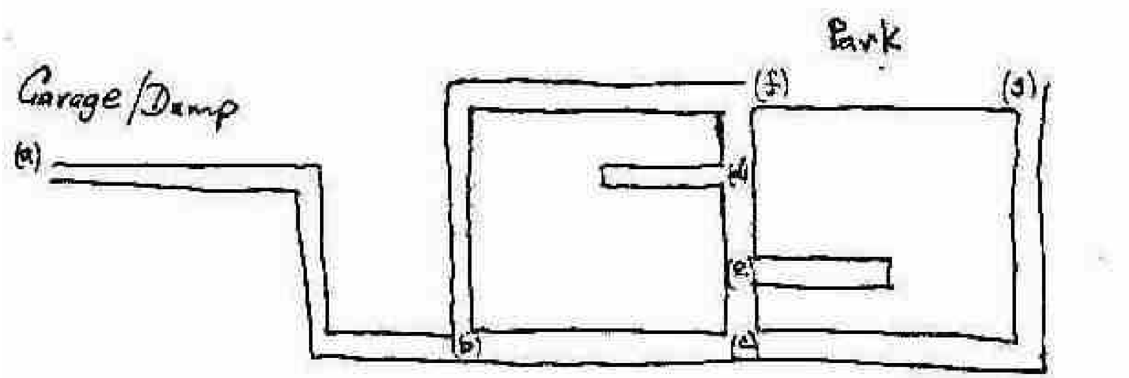
Consider the following simple examples:

(a) Garage/Dump at the same location - all nodes even (no Park).
(b) Garage/Dump at the same location - with a Park bordering one side of one street.
(c) Garage/Dump at different locations – with the only odd order nodes being the terminal nodes: the streets from Garage and the garbage dump to the rest of the network requiring garbage to be picked up on one side only.
(d) Garage/Dump at different locations – each terminal node of even order, needing to pick up garbage from both sides of the street.

(a) Garage/Dump at same location.

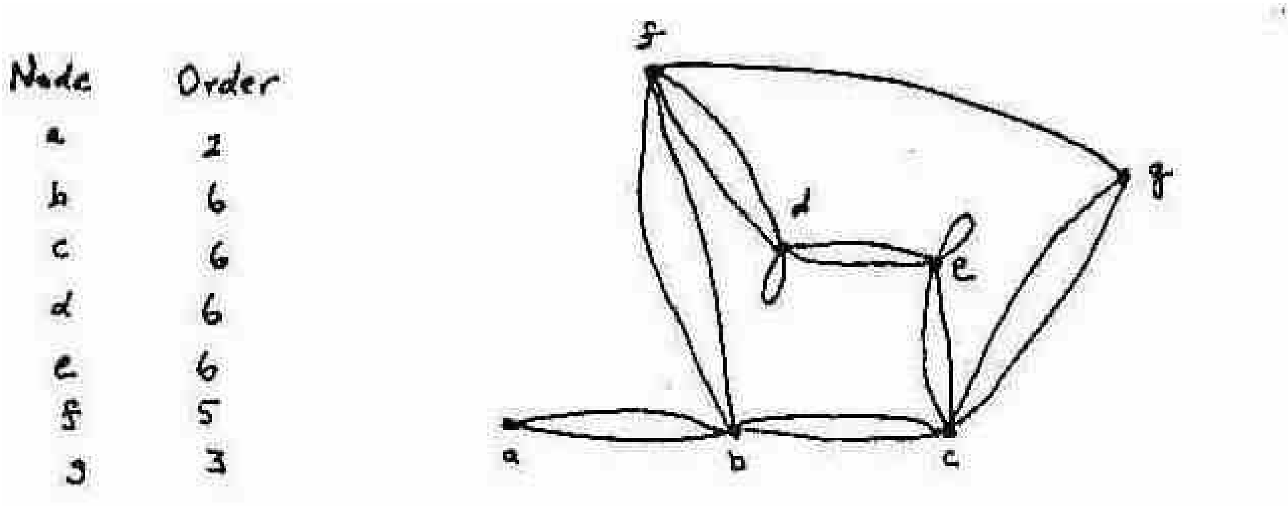
Solution the same as for case above.

(b) as for (a) but with Park on one side of street. This is pictured as follows:



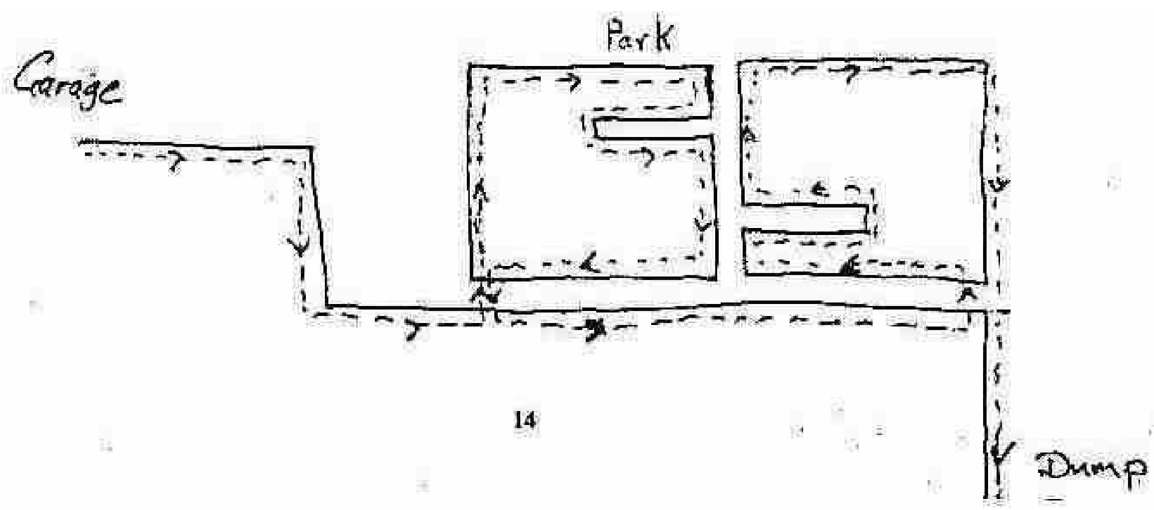
There is no solution without double-back.

The Graph structure of this arrangement is made more obvious from the following dia-gram. The Graph has two nodes of odd order. Hence, by Theorem 2, a continuous path linking all the nodes exists. However, as this path must begin and end at the two nodes of odd order, and neither of these is a terminal node, no solution for routing a truck exists without a double-back.

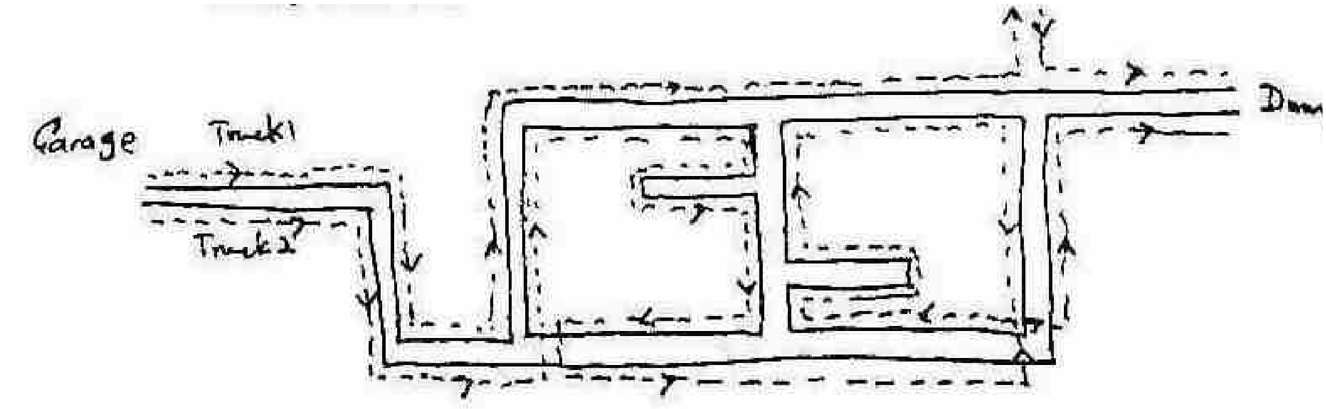


(c) Garage/Dump at different locations.

Two terminal nodes with a Park bordering streets: solution with one truck.



(d) Two terminal nodes of even order: – solution with two trucks, one principally covering another area.



What is the philosophical moral of this story? In the above kind of analysis of a city street-scape, we look at the latter as a large network or graph - treating each street intersection as *a node* and each side of the street between intersections as *a distinct arc*. The orders (or valencies) of the respective nodes are then simply the total number of arcs linked into that node, including the possibility of an arc linking back upon itself to the same node. Thus, in dealing with the practical Operational Research problem of optimally finding routes for the Garbage Trucks, we may draw upon or *apply* the Pure Mathematics of Graph Theory. Hence, we might advance the following thesis:

Thesis Three

The above example exemplifies the relation of 'Pure Mathematics' to the empirical world of practical experience in three parts:

1 *The 'Pure Mathematics' of Graph Theory may be applied to such practical problems as helping to design efficient routes for Garbage Trucks;*

2. *This illustrates the more general thesis whereby Pure Mathematics as systems of thought that, although not needing to be concerned about their reality or practicality when pursued 'for their own sake', yet often have unexpected applications to applied problems;*

3. *In their development, the systems of Pure Mathematics are not concerned with empirical states of affairs. They are concerned with theorems concerning the relationships between entities that are deemed either to 'have an eternal existence in a Platonic World' or in a 'semi-eternal existence in a Humanly Constructed World' - whose ontological relationship to the actual world of daily experience need not concern us.*

There is much within this overall thesis that I do not want to quarrel with. My major quarrel is with the continuing legacy of the epistemological rationalism and/or ontological idealism that would want to claim that the truths of "Pure Mathematics" are either prior to empirical experience or else concern an ideal world that is deemed either to have either a prior Platonic ontological reality to human work or else is to be the result of human construction having no necessary connection with the actual world of our experience.

The falsity of the foundationist features of Part 3 of Thesis Three are exposed and criticised principally by means of the case study mentioned in the title*: Euler and the Konigsberg Bridges*. As such they will be entirely supportive of the anti-foundationist claims as set out in Point 1 of Thesis One. At the same time however, Euler's paper solves an 'empirical problem' every bit as much developing a mathematical theory within which any similar empirical problem might be solved. This, along with the sophisticated 'abstractionist' methodology developed by Euler, provides a strong counterexample to the generality of Part 2 of Thesis One.

3.2. Euler and the Konigsberg Bridges.

The Konigsberg Bridges problem may be stated in Euler's own (translated) words, as follows:

> *In the town of Konigsberg in Prussia there is an island A, called "Kneiphof", with the two branches of the river (Pregel) flowing around it , as shown in Figure 1. There are seven bridges, a, b, c, d, e, f and g, crossing the two branches. The question is whether a person can plan a walk in such a way that he will cross each of these bridges once but not more than once. I was told that while some denied the possibility of doing this and others were in doubt, there were none who maintained that it was actually possible.*

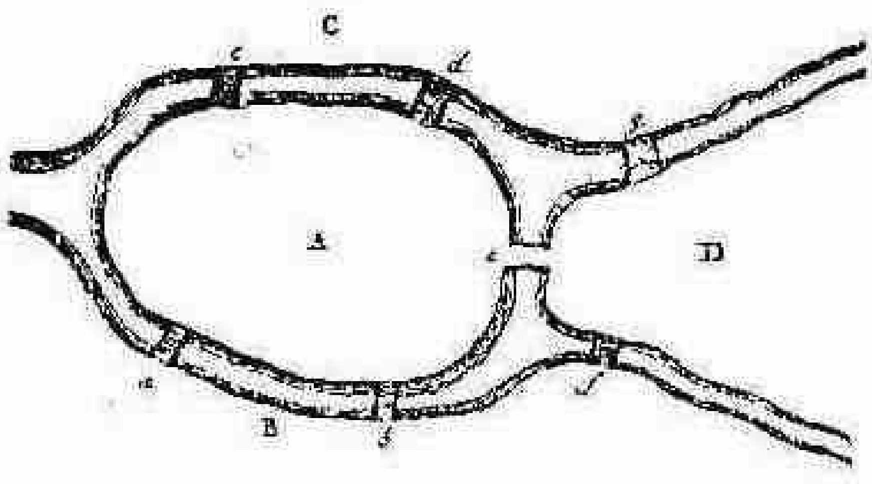


FIGURE 1

In the attempt to find a continuous path, there are obvious parallels with our garbage collection problem. However, there are also some significant differences: one of these is that the intersection of roads occurs in "points", whereas paths across bridges connect sections of land separated by rivers – what contemporary mathematicians would probably describe as topologically connected regions. Whilst the latter are not generally con-sidered 'geometrical points', they may be considered the "nodes" or "vertices" of a Graph just as readily as the "points" of intersection of a road network.

However, the major difference lies in the historical context concerning the possibility of 'applying' Pure Mathematics to practical problems. Euler was unable to 'apply' the theory of graphs to the solution of the Konigsberg problem simply because the theory didn't exist at the time. Indeed, perhaps the main historical significance of the paper is in its contribution to the very development of Graph Theory itself.

Because Graph Theory is generally recognised as a branch of "Pure Mathematics", I believe that it is of some significance that we appreciate the actual steps involved in its development. In this respect we shall proceed as follows:

1. Seek to understand the basic outline of Euler's paper.
2. Seek to appreciate the moves that are required for Euler's contribution to develop into Graph Theory.

1*: An Analysis of the Basic Steps in Euler's Paper:*

I would suggest that we may analyse these steps as follows:

1.1 The Actual Concrete Problem

We have already summarised this in Euler's words above.

1.2 The stripping away of irrelevant details of the concrete setting in respect of this problem.

In the introductory paragraph to his paper, Euler writes:

> *The branch of geometry that deals with magnitudes has been zealously studied throughout the past, but there is another branch that has been almost unknown up to now; Leibniz spoke of it first, calling it the "geometry of position" (geometria situs). This branch of geometry deals with relations dependent on position alone, and investigates the properties of position; it does not take magnitudes into consideration, nor does it involve calculation with quantities. ..... In this paper I shall give an account of the method that I discovered for solving this type of problem, which may serve as an example of the geometry of position.*

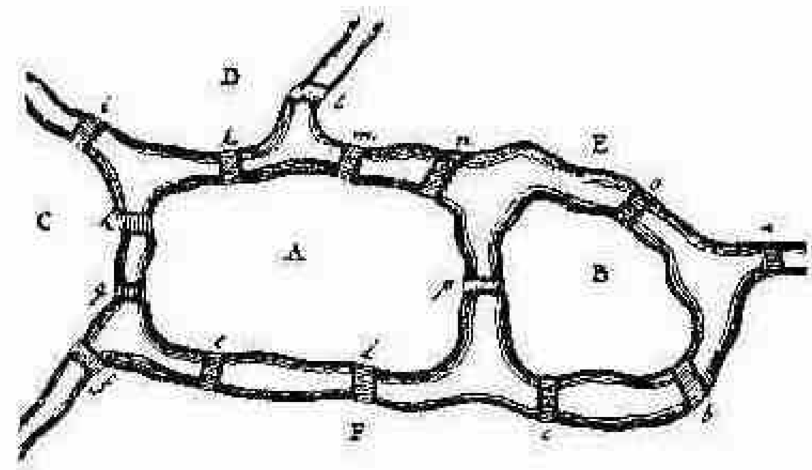
In this respect, well before people had put forward the idea of non-Euclidean geometries (the Konigsberg paper is dated 1735), Euler speaks of a branch of geometry that has nothing to do with lengths, the parallelism of straight lines, or the other hallmarks of Euclidean Geometry! It is not that the streets and bridges of Konigsberg were deemed to be in some fantasy-land without length(!!); it is simply that these details were not considered by Euler to be relevant to an effective means of solving the kind of problem presented by the Konigsberg Bridges. Hence Euler simply stripped away all those details of the real situation that he deemed irrelevant to a solution of the kind of problem at hand.

1.3 The imbedding of the problem in a wide range of similar problems.

Following on from the portion of the second section of Euler's paper concerning the specifics of the Konigsberg Bridges problem, Euler goes on to write:

> *On the basis of the above I formulated the following general problem for myself: Given any configuration of the river and the branches into which it may divide, as well as any number of bridges, to determine whether or not it is possible to cross each bridge exactly once.*

Thus, in amongst the whole variety of possible such configurations, Euler considers the following specific *possible* but not *actual* configuration:



The level of generality envisaged by Euler does not take us beyond the specific type of problem he has introduced. There is no hint of its possible connection with the problems associated with collecting garbage or with more abstract structures. We are simply concerned with the general problem as to the conditions under which a continuous path connecting all the land areas divided by the rivers, utilising each and every bridge only once. Nonetheless, the level of generality introduced into his considerations is vital to his method of solution. In particular, this level of generality takes us into a realm in which we are not restricted to similar *empirical* states of affairs to the Konigsberg case. All possible such similar states of affairs are under consideration, whether or not they are represented by actual empirical states of affairs.

1.4 The development of a mode of representation that aided the solution to these problems.

In principle it would be possible to try to solve this class of problems by carefully tabulating all the options. In respect to this possibility Euler, anticipating the issues of computational complexity such as the 'travelling salesman' problem, remarks that even for the relatively small Konigsberg case, the number of such options would be large, making the problem too tedious and difficult. He then goes on to say that

> *My entire method rests on the appropriate and convenient way in which I denote the crossing of the bridges, in that I use capital letters, A, B, C, D, to designate the various land areas that are separated from one another by the river.*

By using this notation Euler is able to describe a continuous path from one land area to another by a bridge that is denoted by means of the notation introduced for the land areas. From this, two things follow:

(a) (Lemma 1). A path across *n* bridges will require *n+1* letters to denote it.
(b) (Lemma 2). If the number of bridges connected to a given land area, **X**, is odd, say 2 *i*-1, then any continuous path crossing each of these bridges only once will have **X** in the sequence of letters denoting the path across the bridges to the land areas exactly *i* times. Moreover, the minimum path involving all such X's will either begin with an **X** and end with a non-**X** or else begin with a non-**X** and end with an **X**.

1.5 An application of the Pappusian method of Analysis and Synthesis.

The remainder of the paper largely comprises an application of the Pappusian method of analysis and synthesis. Thus Euler sets out the required synthesis in the following terms:

> *Our question is now reduced to whether from the four letters A, B, C, and D a series of eight letters can be formed in which all the combinations just mentioned occur the required number of times.*

However, before endeavouring to do this, he considers the possibility of the method of analysis:

> *Before making the effort, however, of trying to find such an arrangement we do well to consider whether its existence is even theoretically possible or not. For if it could be shown that such an arrangement is in fact impossible, then the effort expended on finding it would be wasted.*

The method of analysis involves the assumption of the truth of the conjecture under consideration and the drawing of logical consequences from it. The conjecture in question here is the existence of a continuous path over each and every bridge only once.

Since, in the case of the Konigsberg Bridges, the number of bridges connecting each land area is odd, it follows from Lemma 2 above that :

A must occur **three** times in the resultant sequence of letters (5 bridges connect to A);
B must occur **twice** in the resultant sequence of letters (3 bridges connect to B);
C must occur **twice** in the resultant sequence of letters (3 bridges connect to C);
D must also occur **twice** in the resultant sequence of letters (3 bridges connect to D).

In other words, there must be **a minimum of nine (3+2+2+2) letters in the sequence**.

However, from Lemma 1, since there are seven bridges in all, a sequence of letters denoting a continuous path across each bridge once and only once **requires no more than eight letters**.

This classic contradiction is sufficient to establish the result that there is no way in which the seven Konigsberg Bridges can be crossed by means of a continuous path that crosses each and every one of them only once. Moreover, in respect of the general case in which all the land areas have an odd number of bridges leading onto them Euler makes the claim that Lemmas 1 and 2 are sufficient to establish the synthesis condition as to whether or not a continuous path crossing each and every bridge only once exists:

If there are K bridges and n land areas: $L_1, L_2, ...L_n$ each with $j_1, j_2, ...j_n$ bridges leading to or from them, then if each of $j_1, j_2, ...j_n$ are odd ($j_k = 2\,i_k - 1$; $k = 1....n$) such a path will not exist if

$$i_1 + i_2 + \quad + i_n > K+1,$$

as in the case of the Konigsberg Bridges. Moreover, the replacement of the inequality by an equality in this relation is sufficient for the required continuous path to exist. However, these results are really part of the general synthesis argument that seeks to establish sufficiency conditions for the existence of a continuous path through each and every node only once. For this purpose, Euler seeks to establish the following results that are here stated in the form of Lemmas 3,4 and 5.

Lemma 3:

If the number of bridges onto a region A is even (say $2i$) then

(i) If a path is commenced at A, then
(a) the number of times A occurs in the sequence of letters denoting the minimum continuous path including all the visits to A is $i+1$.
(b) the path begins and ends with A.

(ii) If a path is commenced in some other region, then
(a) the number of times A occurs in the sequence of letters denoting the minimum continuous path including all the visits to A is i.
(b) the path begins and ends with non-A.

Lemma 4:

If there are K bridges and n land areas: $L_1, L_2, ...L_n$ each with $j_1, j_2, ...j_n$ bridges leading to or from them, then, since each bridge connects exactly two land areas,

$$j_1 + j_2 + \quad + j_n = 2K,$$

from which it follows immediately that only an even number of $j_1, j_2, ...j_n$ can be odd.

Lemma 5:

If there are K bridges and n land areas: $L_1, L_2, ...L_n$, with L_k occurring p_k times in the sequence denoting the continuous path over each and every bridge only once, then the necessary and sufficient conditions for the existence of a such a path is:

$$p_1 + p_2 + \quad + p_n = K+1$$

Suffice to say that Euler completes his paper by establishing results equivalent to those of Theorems 1 and 2 for Graph Theory and subsequently applied to the Garbage collection problem stated in the first section of this paper.

Theorem 1 follows from Lemmas 3,4 and 5.

$p_1 = i_1 + 1;\ p_k = i_k \quad (k=2....n)$ from *Lemma 3*

$j_1 + j_2 + \quad + j_n = 2K$ from *Lemma 4,* whence the result for *Lemma 5* follows

Theorem 2 follows similarly.

Euler concludes his paper with some remarks about the finding of an actual continuous path over a set of bridges when such a path exists. His theorems do not yield such results directly. The areas of land are then designated by letters. These letters are then used in pairs to designate bridges across the rivers, losing the ability to distinguish between individual bridges between the same two land areas in the process. In a complex problem, with many bridges between the same two land areas, the use of Theorems 1 and 2 may help to find whether or not a solution to the problem exists. In the case of the Konigsberg Bridges, the whole problem is settled relatively quickly. In the case in which a solution exists he goes on to suggest a method whereby such a continuous path may be found – very often the solution won't be unique. This technique simply entails the mental elimination of pairs of bridges between the same two regions. The pattern of a solution across the remaining bridges may then usually be found – especially in the case of smaller problems. A complete solution may then readily be found by reincorporating the previously eliminated bridges into the particular solution obtained – *'as a little thought will show'*.

This completes the survey of Euler's paper. Our next step is to investigate what is involved with the development of Graph Theory from Euler's work.

2 *The Move to Graph Theory*

In the move to Graph Theory, the letters or the areas designated by them become ***nodes*** **or** ***vertices***, whilst the continuous connections between the nodes become ***the arcs***.

This move enables both the street-scape of the garbage collection problem, the Konigsberg Bridges problem as well as any other concrete problem possessing this kind of structure to be considered a graph. Moreover, it is a simple matter to adapt Euler's arguments – which, as given by him, have the more specific concrete focus of land/ rivers/bridges - to the more abstract one of Graph Theory. Moreover, because the street-scape and the land/rivers/bridges – as well as any other concrete structure – exhibiting these features - share the basic Graph structure, the results concerning the odd/even orders of the nodes of the Graph for the continuous paths linking all the nodes once and only once, apply to them.

I would now like to return to **Thesis Three** concerning the relationship between Pure Mathematics and the empirical reality of the world of our experience.

Parts 1 and 2 of this thesis do not need to come under the hammer directly. However the details of Euler's contribution to the pure mathematics of Graph Theory clearly expose the falsity of the general claim as to the way in which Pure Mathematics develops – both in relation to the practical problems arising from daily empirical experience, as well as in respect of the way in which mathematical problems and structures are **applied** to empirical states of affairs.

Euler's 'The Seven Bridges of Konigsberg' starts from a concrete practical problem, and proceeds to develop a theory of similar such problems. As such this work contributed to the growth of empirical knowledge – the good folk of Konigsberg could abandon their search for a continuous path comprising each and every bridge over the Pregel only once.
Who knows, this may have been the reason why one of the more prominent sons of the city over the next century was able to continue his daily walks with his mind taken up with other things – I'm referring, of course, to none other than Immanuel Kant. How-ever, one suspects that the obvious empirical nature of the geometrical knowledge con-tained in Euler's paper would have caused some serious problems for Kant's theory of knowledge and that he was blissfully ignorant of the significance of Euler's achieve-ment.

Furthermore, it is clear that the more abstract, 'pure' character of Graph Theory actually grew out of **an application** of the general but less abstract character of 'the geometry of position' problems applied to the 'Land/Bridges/River' theory of Euler.

Thesis Four

In the light of the consideration of Euler's paper, I would like to replace Thesis Three by Thesis Four as a general conjecture concerning the way in which Pure Mathematics develops and relates to the empirical problems of the concrete everyday world.

1. In the first place Pure Mathematics comes into being as the result of the consideration of concrete problems that arise from 'World 1' - the world of daily experience, illustrated by the Konigsberg Bridges.

2. In the process of their development such theories involve:

   A process of abstraction, whereby a whole range of properties present in the concrete situation are (mentally) stripped away from consideration, thereby **isolating certain properties for particular attention**.

   A process **of definition** whereby these particular properties under consideration gain a precision that is not characteristic of common-sense experience. This usually involves several concepts – some abstract, some possibly concrete – in **the formation of a theory.**

   A process **of generalization** in which the specifics of the particular problem under consideration are imbedded in a wider class of problems.

   An application of the method of **analysis** (the discovery of likely conjectures as well as their possible refutation) and **synthesis** (proving theorems) in respect of the kinds of problems considered – illustrated further by the work of Imre Lakatos in *Proofs and Refutations*.

3. Once developed, theories usually begin to take upon a life of their own. In this way results may indeed be developed that have no immediate application, but because of their background in respect of their links with concrete reality via the above kind of process, may be expected to find (further) application to empirical problems. However, it is a mistake to think that this 'life of their own' character of Pure Mathematical Theories develops independently of the common-sense experience of empirical reality.

4. The example of the application of Graph Theory to the problem of designing routes for garbage trucks considered at the outset of the paper, is illustrative of this process:

   - Graph Theory grew out of the ‘Bridges/Land/Rivers’ theory of Euler.
   - Many features of the actual structure of Road Networks are reflected in the abstract structure of Graph Theory.
   - Hence the results of Euler’s contribution to the *geometria situs* of ‘Bridges/Land/Rivers’ may be expected to have some application in respect of those features of actual Road Networks that have a similar mathematical structure.

4 Mathematics and Culture.

4.1 Mathematics is a Human Cultural Activity.

Mathematics is a human activity. This is clearly illustrated by the case of Euler’s Konigsberg Bridges. Moreover, the case is also illustrative of the two primary activities involved with mathematical development:

- Problem solving;
- Theory construction.

In respect of problem solving, the case further illustrates a distinction between concrete problems and hypothetical problems. Euler showed that it was not possible to find a continuous path crossing all of the Konigsberg bridges once and only once. In this respect his achievement made a contribution to empirical knowledge. He also showed that this task **was** possible for a particular hypothetical configuration of bridges/rivers/ land areas that, at least as far as he knew, had no actual concrete existence. Much of mathematics falls into the latter category. Moreover much of this can be useful in the design of engineering and other industrial systems: the major features of the hypothetical problem may be solved before it is concretised.

In respect of theory development, the case illustrates the actual genesis of a new theory very well. Along with a number of other seminal papers, Euler’s contribution was influential in the development of two major fields of modern mathematics: graph theory and topology. The concrete problem background together with the more important innovative contributions to the these theories are well illustrated in the book *Graph Theory: 1736-1936* by Biggs, Lloyd and Wilson (Oxford University Press, 1976).

As this case is typical of a large part of mathematical activity, mathematics is clearly a field of human culture that involves human social interaction in the manner claimed by Hersh in Part I of **Thesis One**. The next main step within the present paper is to give attention to Part II of that Thesis.

In this respect we can distinguish between three types of mathematical activity in respect of theory development together, each of which involves general issues of method:

- The genesis of a theory;
- The internal development of a theory;
- The application of a theory.
- Methodolgy.

4.1.1 The Genesis of a Theory.

This activity is well illustrated by the case study of Euler's Bridges. As such the following features may be said to be typical of what is involved:

(i) *The actual concrete problem.*
(ii) *The stripping away of irrelevant concrete details in respect of this problem, so as to isolate the relevant properties to be investigated.*
(iii) *Imbedding of the problem in a wider range of similar problems.*
(iv) *The formulation of a theory involving abstract concepts that include precise formulations of the properties isolated in step 2. These definitions are fallible human formulations having conjectural features that may not directly evident to the concrete situation.*

4.1.2 The Internal Development of a Theory.

This activity is not well illustrated by the Konisberg Bridges problem. However, it is illustrated by other developments within Graph Theory and Topology, as well as almost every other field of mathematical endeavour – such as those of number theory and integration theory mentioned below:

(i) *A theoretical problem within a theory. Examples might be the conjecture concerning Fermat's Last Theorem or the problem of the non-existence of the Riemann Integral of a real-valued function (such as that for which the value is 1 for X rational and 0 for X irrational) over the interval [0, 1].*
(ii) *The existence of such problems frequently do not have any particular concrete focus. The latter are abstract in character.*
(iii) *Such problems can also lead to significant developments in a theory – exemplified by the recent proof of Fermat's Last Theorem and the development of the Lebesgue Theory of Integration and Measure Theory.*

4.1.3 The Applications of a Theory.

This activity is well illustrated by the case study in a way that includes both the Garbage Truck problem as well as the application of Euler's theory to the Konigsberg Bridges. In particular the application of Graph Theory to the solution of the Network Garbage Problem is illustrative of what is entailed in the application of mathematical theories to the concrete 'World 1'.

(i) *Either a practical concrete problem or an abstract problem in a neighbouring theoretical field may exist – requiring a solution.*
(ii) *Known existing results from a theory may be applicable to either of these types of problem. As such the known solutions may be 'applied' to obtain a solution to the new problem.*

4.1.4 Methodology

Without going into detail we may characterise the methods of both mathematical problem solving and theory development in each of these three types of activity by:

Trials, Conjectures, Computations, Refutations and Proofs.

In respect of the Konigsberg Bridge problem, much of the work done in respect of trials and conjectures done by Euler was either done in his head or else on scraps of paper. In any case they were not included in his final presentation. The methods employed in it were those of the classical Pappusian analysis and synthesis. These matters have been both further developed and well explored by both Georg Polya and by his student Imre Lakatos, and have been well illustrated for simple cases in the earlier writings of Hersh and Davis – consider the simple example in 'The Creation of New Mathematics: an application of the Lakatos Heuristic' (in *The Mathematical Experience* pp291-298).

4.2 What distinguishes Mathematical Socio-Cultural activity?

Now, we may readily identify Hersh's 'socio-cultural-historic' view of mathematics in each of the three types of activity associated with theory development and problem solving mentioned above. However, human life comprises many other such activities – engineering, commerce, sculpting to name but a few. To be able to give a credible answer to the question *What is Mathematics Really?* requires that an equally important question that needs to be addressed is: 'What distinguishes *mathematics* as a social-cultural activity from *some other* such activity - such as sculpting?' Granted, those organising mathematical activities – such as conferences, journals and university courses - and sculpting activities – such as art classes, exhibitions and purchases of finished works - already have a pretty good idea – but these ideas are largely rooted in the intuitions of what Polanyi would call 'tacit knowledge'. As such, this kind of knowledge is very often lacking in clarity and theoretical/ philosophic precision. Surely one of the important tasks of philosophy is to clarify this tacit knowledge - in the sense of endeavouring to give a fallible theoretical account of it. In this sense, Hersh, in the preface to his book, after criticising the classic *What is Mathematics?* by Courant and Robbins comments in respect of the latter:

> They never answered their question; or rather, they answered it by *showing* what mathematics is, not by *telling* what it is. After devouring the book with wonder and delight, I was left asking, "But what is mathematics, really?" (p xi)

After making an excellent start in articulating the view that mathematics is part of the broader world of human cultural objects, Hersh seems to want to dogmatically cut off any further attempt to clarify just how the human activities of mathematics and sculpture both produce cultural products, but of very different sorts. In other words not only do we need to be able to say what it is they have in common, we also have to be able to try to articulate with philosophical clarity just how they differ. I, for one, am left with a similar sort of reaction to his book that he experienced with his reading of Courant and Robbins.

In most, if not all respects, it would seem to me that Hersh's discussion of mathematics as a 'socio-cultural-historical' phenomenon is most appropriate to what I have called above *the internal development of theory.* In this part of mathematical activity it is possible to focus upon internal matters, avoiding the important issue as to the reasons why mathematical theories apply to practical problems as well as that having to do with just what the theories are all about anyway!

Thesis Five:

The attempt to correct Hersh on this point would seem to require answers to the following two questions:

1. What is the demarcation between theories/contents of thought from *other* cultural objects?

2. What are the distinguishing marks of mathematical theories?

An attempt will be made to address both of these questions in the remaining sections of this paper. In doing so we will draw heavily from the case study of *Euler and the Konigsberg Bridges.*

4.3 The Place of Theories in 'the World of Cultural Objects'.

To get from one side of a deep, wide river to the other generally requires the kind of cultural formative activity that results in the use of a barge or a bridge. Euler's attempt to answer the question as to the possibility of a continuous walk over all of the seven Konigsberg Bridges without crossing any bridge more than once was achieved via the construction of a mathematical theory. Whilst both the barge and the mathematical theory constructed by Euler are 'cultural objects', quite clearly they differ in their ontological status. Whilst both may be 'weighty', once learnt, Euler's theory is probably a lot easier for one person to carry around!

Quite obviously a theory is a set of concepts. As such it exists primarily within 'World 2', having been thought out and developed either by one person or by a group of persons in social interaction. Moreover, even in the former case there is usually a social context to the problem situation that the theory seeks to address. Thus the objective character of the theory is articulated in symbols and language. This means that once Platonism is rejected, theories have a somewhat precarious form of existence - they have an objective existence in the form of symbols and language that point to concepts that are humanly formed. There is always the question as to what these concepts are, what they mean, and how they relate to the concrete entities of 'World 1'. For example, to say that Plato's philosophical ontological theory is a cultural object is not a matter that can be disputed. The dispute is all about whether or not it refers to anything beyond the concrete entities of 'World 1'. In other words all the 'mainstream' philosopher of mathematics has to do to avoid Hersh's critique is to admit that the total heritage of Platonism – Foundationism is simply a 'socio-cultural-historic' activity, and then to continue on doing it as if nothing had changed!

A theory is a cultural object comprising concepts symbolised in language. This makes it different from a barge or a bridge. However, a theory may be scientific, metaphysical, philosophic, speculative, tenuous, well-attested and many other things. Which of the possible epithets to be applied depends upon the purported relationship between the set of concepts in 'World 2' (and symbolically objectified in 'World 1') to the realities of 'World 1'. In the case of mathematical theories the particular types of activity that impinge most directly upon this issue are *the genesis of the theory* and *the application of the theory* as discussed above. In these respects the case study of Euler's bridges has some significant implications for a discussion of this matter.

The relationship between the Konigsberg Bridges and Euler's theory has been examined in detail within section 3. A key step in this relationship was the role played by *abstraction.* Hersh's discussion of the role of abstraction in the development of mathematical theories is ambiguous. On the one hand he recognises its significance; on the other he does not think that it can be developed beyond elementary mathematics. In his discussion of the predecessors of his philosophical heritage for example, Hersh makes reference to Aristotle and his 'abstractionist approach to Mathematics' as follows:

> *In Aristotle's philosophy of mathematics, the key concept is abstraction. Numbers and geometrical figures are abstracted from physical objects by setting aside irrelevant properties – colour, location, price, etc – until nothing's left but size and shape (in the case of geometric figures) or "numerosity" (in the case of finite sets). As an account of elementary mathematics, this is not bad. Today it's inadequate, because mathematics includes much more than circles , triangles, and the counting numbers. His account is clear and reasonable. But by twentieth – century standards it's not precise. It would be difficult to give a formal definition of abstraction.* (p183)

Granted - the account of abstraction given by Aristotle is inadequate to meet contemporary standards. However, that does not mean that an adequate account cannot be given. Not only do I intend to try to do that, I also wish to claim that the case study of this paper gives some very strong support to an account of mathematical activity that includes this feature as a principle element in what is involved with the actual practice of mathematics. To establish this let us look again at the way in which Euler's work exemplifies the process of abstraction as a significant feature of mathematical development. I would suggest that there are three steps of the abstraction process that may be clearly identified in his Konigsberg Bridges paper.

4.3.1 Stripping away a whole range of property types from the concrete reality.

The problem confronting the good people of Konigsberg was concrete. As such the properties of the diverse things making up the world of their experience were many and various. The first step in Euler's analysis was to mentally ignore or strip way all those properties of the concrete things that were not pertinent to a consideration of his problem. This step resulted in the geometrical map/diagram of the land areas/river/ bridges presented in his paper. As a step in the process of abstraction it is almost so obvious as not to require mention. It is the kind of step in the process of abstraction envisaged by Aristotle himself.

4.3.2 The Stripping Away of all Considerations of Length within the already Abstracted Geometrical Picture.

The second step in the process is to strip away all considerations of length in respect of the geometrical situation envisaged by the map/diagram of the rivers, bridges and land areas of Konigsberg. This step in its time was actually very innovative. It may not seem so to us, but then we live after the many innovative developments of non-Euclidean geometries/ projective geometries and topologies developed during the nineteenth and twentieth centuries.

Indeed, I put forward the suggestion that it was precisely this step of abstraction that has led to the historical development of both Graph Theory and Topology since the time of Euler's paper.

4.3.3 The Land Areas as Nodes or Vertices

Euler's actual next step was to develop a notation, not one of abstraction. However, his contribution in this respect is suggestive of the further step in the process of abstraction that would be characteristic of a modern treatment of the problem. He **denoted** the land areas by **letters**. In this respect he still conceived of his abstractions as linked with land areas – what would be referred to today as *a connected topological sets*. In this respect others were later able to further abstract or remove the specific content of a connected topological region to be left with a *node* or *vertex* that might be instantiated as a topologically connected region or a point, and with concrete, empirical reality as land areas or street intersections.

This account is clearly sufficient to refute the general claim made by Hersh concerning the impossibility of the Aristotelean tradition of abstraction being developed so as to be able to account for many of the significant features of modern mathematics.

4.4 The Ontological Status of Abstract Concepts

Abstract concepts are produced by human work. As such this involves 'World 2' activities in their interaction with 'World 1'. There are at least two major steps in this *process of abstraction*. In the first certain properties of 'World 1' entities are stripped away with a view to gaining a greater clarity of insight upon those properties that are left. In this respect we may speak of *isolating* the said properties. A geometrical line is an example of such an isolated property that results from the (intentional or mental) stripping away of all the other properties of 'World 1' entities till only the line, complete with a lack of chalk marks and width etc, is left. Of course it is impossible to draw a picture of this line – simply because it has had all the physical properties removed by the mental act of isolation. The second step entails the formation of a concept of this property, usually associating it with a range of other concepts – both abstract and concrete – so that tentative theories are formed as contents of thought within 'World 2', and symbolised as cultural objects in 'World 1'. As such they have the potential to apply to a wide range of 'World 1' entities, precisely because they have arisen from this world in the first place. The abstract concepts in theories relate to the concrete entities of 'World 1' by virtue of the way in which the isolated properties, formulated in the theory, are instantiated in these entities. The theoretical formulation of such concepts is fallible human work, and usually builds upon the work of others. However, it seeks to give an account of the regularities of 'World 1' events and processes, focussing upon a range of particular properties. Very often the latter are then viewed through 'the lens' of the theory.

The steps in this kind of process - involving the development of abstract concepts by World 2' activities upon 'World 1' entities - plays a significant part in all theorising. Hence, the question concerning their ontological status needs to be addressed. We may distinguish four main philosophical traditions in this respect:

The Platonic Tradition, in which the abstract concepts are accorded an eternal ideal existence, as **forms** – in terms of which the corresponding concrete properties of the sensible world – World 1 – are but a pale reflection. The implications for mathematical objects are obvious and has already been discussed extensively.

The Cartesian- Kantian tradition – in which the abstract concepts arise as part of the rational structure of World 1 – as 'a world comprising *primary* properties' as known by inhabitants of World 2. However, much of this structure is deemed to arise from the contribution of the conceptualisations imposed by the mind of the World 2 inhabitants. Furthermore, the issue of certainty and logical coherence entailed with this tradition is a prime factor in the foundationist theories that have dominated the Philosophy of Mathematics over this century.

The Empiricist reaction to the extreme claims of the Platonist view of abstraction has been one that would seek to deny any ontological reality to the entities connoted by abstract concepts. Twoness, for example, only exists in relation to concrete entities. J. S. Mill is the obvious candidate for this view in respect of Mathematics.

The Aristotelian tradition, in which abstractions are firmly founded in the properties of actual entities within 'World 1' but, as such , may be conceptualised and function within the body of theories that have objective contents by virtue of being founded in the properties of entities within 'World 1'. In Aristotle's actual philosophical system the abstractions were also identified as **forms**, a feature that gave them the character of a certain metaphysical theory. However, in contrast to the Platonic system, no ontological status was accorded to the forms other than that realised in the 'World 1' objects themselves.

The view of abstract concepts developed in this paper takes its cue from the Aristotelean tradition of abstraction. At the same time, however, it (i) carefully eschews any identification with forms, and (ii) emphasises the role played by the exercise of human formative power in respect of the development of the particular abstract concepts of the properties isolated in the process of abstraction. This renders the general process of theory format-ion fallible and open to rival interpretations. In suggesting the view that abstract concepts arise from the intentional mental activity of human beings does not imply that they are arbitrary creations. On the contrary they have a (conjectured) real existence in 'World 1' entities by virtue of their relation to the isolated properties they are attempting to conceptualise. The latter properties exist in concrete things/ processes/ events; the theory relates to these properties via the set of abstract concepts that are formulated by it.

It should be noted that a theory generally comprises a number of abstract concepts, each having a dependence upon the other. Many theories do not compete. They result from the formulation of quite different sets of *properties* from the concrete things/processes/events of 'World 1'. Theories compete when they entail different theoretical abstract concept formations of similar properties.

In ordinary, concrete experience we focus upon 'the things in coherence with their properties', whereas when we engage in the theorising involving abstract concepts, we focus upon 'properties as they are instantiated in concrete things'. However, the way in which these properties are instantiated in things, now carries the freight of the hypothesised theoretical abstract concept formation. However, no abstract entities having some independent form of existence in the vain of Platonism, is involved.

To summarise, theoretical abstract concepts arise from a process of abstraction that embodies at least two main features:

- The (mental) stripping away of irrelevant properties form concrete reality, thereby isolating certain properties occurring in a range of concrete things.

- A process of theoretical concept formation that seeks to formulate precise concepts of these isolated properties, so that the former may be viewed as 'instantiated' in concrete things.

Newton's Theories

We may briefly illustrate this process with reference to the development of Newton's theories of motion and gravitation. The well known laws of motion are as follows:

N1. Every **body** continues in its state of rest or **uniform motion in a straight line** unless acted upon by an external **force.**
N2. The net external **force** acting upon a **body** is proportional to **the rate of change of momentum of the body** and is effected in the same direction as that of the resultant external force acting upon it.
N3. To every **action** there is an equal and opposite **reaction**.

The concepts of body, force, uniform motion in a straight line, momentum, action, rate of change all result from the kind of process of theoretical abstraction discussed above.

(i) The term 'body' is used with no specific reference to a living or dead body. Indeed such properties are irrelevant. Not only the life/dead properties, but also such properties as aesthetic and economic, as well as those of human cultural formative power are abstracted from the concrete things of 'World1' by mental activity so as to isolate the physical properties that are deemed relevant to the motion of stones, birds, canon balls and planets. In this process of abstraction the tendency of the philosophy of the seventeenth and eighteenth century was to attribute the properties left after this process of abstraction to 'the things in themselves' - the primary properties, whilst attributing the other properties - the secondary properties, to human subjects. The legacy of this tendency lives on, being well illust-rated by the problems already discussed in respect of Popper's World 3 hypothesis.

(ii) It took the exercise of a great deal of human formative power in respect of the precise formulation of the theoretical abstract concepts developed in the theory. In particular the Principle of Inertia, whereby 'the natural state' of motion was postulated as *uniform motion in a straight line* entailed a great deal of work, building upon the work of Galileo and Descartes, in particular. The fundamental effect of Newton's formulation is to provide a test for the action of a force upon a body: whenever a change from uniform motion in a straight line occurs, the body concerned is deemed to be acted upon by a force – whether it be a stone falling to the ground, a feather rising up in the wind, a bird flying up and down over the waves or the potential motion of a flying carpet! The role of the earlier work of Galileo's Terrestrial Kinematics and Kepler's Planetary Laws is well known.

(iii) The exercise of this human formative power upon the precise hypothesised content of these physical and kinematic properties of concrete objects was not developed individually, but upon the group of concepts collectively.

Many theories do not compete. They result from the formulation of quite different sets of *properties* from the concrete things/processes/events of 'World 1'. Theories compete when they entail different theoretical abstract concept formations of similar properties.
In the case of rival theories, the isolated property is the same, but the abstract concept formulation is different, thereby providing the basis for a test of the comparative explanatory merits of the respective theories via the actual properties of concrete things.
This also provides the means whereby the newer theories can actually provide a level of predicted precision that was not present in the older theory. (See Popper's paper 'The Aim of Science', in *Objective Knowledge*)

Thus abstract concepts carry a large amount of human formative effort as they function within the framework of other concepts – both abstract and concrete - in theories. In this sense, I think Popper has the right aspiration in his attempt to identify theories as 'World 3' objects. However I believe that Popper, in his view of abstractions, is in the Cartesian-Kantian tradition rather than the Aristotelean one. He therefore has no place for a process of *abstraction* whereby an act of mental focussing upon certain properties to the deliberate exclusion of others is an integral part of the development of the *abstract concepts* forming the content of humanly constructed theories.

Thus we may say that abstract concepts have the ontological status of *theoretically formulated properties* of the things/processes/events of concrete 'World 1' entities. Hence, while an abstract concept does not have the same status as a property it arises from the latter by the human shaping activity of 'formulating a precise concept of an isolated property' (such as mass, charge, length, node, arc) in the attempt to develop a theory that embodies a number of concepts of such properties. However, this latter process in and of itself does not render the abstract concept any other ontological status. In particular the notion of *abstract entities* – such as numbers, perfect solids, and any other Ideal Platonic entity – having a distinct existence apart from the relationship between isolated properties of concrete 'World 1' entities and their abstract concepts formulated in theories, has no basis whatsoever. The evidence for their existence prior to the process of abstraction is slim if not empty, and the latter process itself certainly does not create them.

The various properties of concrete things/events/processes of 'World 1'in their relation to the abstract concepts of theories formulated within 'World 2' and objectivised in symbolic form as talks, discussions, books, papers, computer disks and libraries set the primary ontological limits of abstract concepts. Of course the objective of theorising is (or should be) that of seeking to (fallibly) understand the ways in which the cosmos is lawfully ordered. As such this raises the ontological issue of the ultimate source of the lawful ordering responsible for the regularities within 'Worlds 1&2' that our theorising seeks to capture. As such the culturally formed beliefs with respect to such religious issues are themselves a class of properties belonging to 'World 1' – one that these days is explored in Religious Studies as well as Theology. As such these point beyond the horizon of 'Worlds 1&2'. In many respects the issue of the ontological status of abstractions, both for the Platonic tradition as well as the others, is illustrative of what is involved with going beyond this horizon. For the broadly 'Aristotelean' vantage point proposed here the process of abstraction deals with properties of actual things/events/ processes, with results that have been already described. The actual philosophical systems that were developed within this tradition, however, have generally embodied notions of lawful ordering that were linked to the ideas of form and matter that went beyond this horizon.

The Platonic tradition also involved these notions of lawful ordering, but, in addition elevated the abstractions to an Eternal Ideal world that is beyond the concrete 'World 1'. In this respect it attributes a prior or independent mental/spiritual existence to the properties of the things/events/processes of empirical concrete experience.

The Cartesian –Kantian tradition, on the other hand, suffers from the mistake of attributing some properties (the primary qualities) to things – whereupon 'World 1' is deemed to have be exclusively *physical* character and whereas others of its properties (secondary qualities) to the mental elaboration of things /entities of 'World 1' by the activities of 'World 2'. In effect this attributes a significant role to the lawful ordering of the cosmos to the human mind! In this respect the breakdown of the more specific set of Kantian categories together with their replacement by other theories – scientific /religious /philosophical etc – is an important factor in the development of post-modernism, particularly in the kind of philosophy of science espoused by Paul Feyerabend. It is in this sense that the legacy of the Cartesian-Kantian tradition in Popper's philosophy may be said to be the intermediary between the Logical Positivists and the post-modernist movement.

Finally, the empiricists are suspicious of abstractions to the point of not wanting to give serious consideration to the way they have contributed to the growth of scientific knowledge, a significant reason why Popper saw the need to develop his 'World 3' hypothesis as a means of trying to locate human knowledge, scientific knowledge in particular, within the world of human cultural objects. However, with his (correct) insight as to the link of his 'World 3' with the Platonic world of forms and the Hegelian world of Absolute Spirit together with his failure to deal with the process of abstraction in theory development implies that there are many ways in which his formulation of 'World 3' remains open to embracing the legacy of the Platonic tradition whereby abstract concepts are given the status of entities in a sense that requires clarification. The claim made here is that only a formulation in the Aristotelean tradition can successfully avoid the ambi-guities of his Platonic/Kantian synthesis albeit ageeably spiced with his own theory of the growth of scientific knowledge.

4.5 Planets, Electrons and Numbers

These issues may be illustrated with reference to the ontological status of electrons, hypothetical planets (such as Neptune and Vulcan) and natural numbers.

The discovery of the planet Neptune grew out of a crisis within the Newtonian theory of mechanics and gravitation – the orbits of the known planets showed some significant deviations from the predictions of the theory. Rather than see the theory refuted by this evidence, a hypothesis was put forward to the effect that there was another planet in the solar system (Neptune) whose relevant properties – mass and orbit - were postulated so as to be able to account for the anomalies. The properties of the planet, as relevant to the theory, were **not** put forward as abstractions. Whilst no mention was made of the 'geo-logy' or the 'geography' of the hypothesised planet, it could realistically be assumed that it had some properties akin to the surface and core of the earth. The nature of these properties could only be discovered by inspection – either directly or else with the aid of instruments such as telescopes. In other words, although the hypothesised entity was put forward for theoretical reasons, it was not itself an abstract entity. Certain properties – those required of it by Newtonian theory – were attributed to it for the purposes of the theory. However, it was also assumed to have a variety of other properties - its 'geology and geography', for example - that were not covered by the theory. The fact that the telescope when pointed to the appropriate position in the sky, revealed a planet, enabled the 'hypothesised Neptune' to become a fact of 'World 1'. As such modern space probes are now able to investigate the many properties it has outside of its role in the Newtonian theory of the solar system. On the other hand, the fate of another 'hypothesised planet', Vulcan – invented to explain the observed precession of the perihelion of Mercury so that it conformed with Newtonian Theory – was different. The hypothesised planet could not be observed and Einstein's theory was able to give a better explanation of the anomaly without Vulcan. The net result is (a) that the ontological status of Vulcan remains within the same realm as Centaurs and (b) that Newton's theory has been superseded by Einstein's.

The electron was also postulated as a concrete thing, not an abstraction deemed to have just those properties – mass and charge - required for it to explain the phenomena of cathode rays. This is notwithstanding the fact that electrons are not visible either to the naked eye or with the aid of a microscope. Who knows, should we ever be able to go on tours of the micro-world, it might be very interesting to see just what other properties that electrons possess or indeed if they exist in the manner proposed by the theory. In the latter case electrons too would have to go the way of centaurs.

All this contrasts with the status of natural numbers. Of all the many particular things of our concrete experience of things/events/processes, numbers do not occur as individual entities. Rather, they arise as properties of concrete entities as these display a unity within multiplicity. Most flowers, for example, have petals. On any individual flower the latter are similar to one another but are nonetheless individuated so that there is an answer to the question 'how many petals are there on this particular flower?' In this sense the *number* of petals on a flower is a property both of the flower and of its set of petals. It is possible to abstract all the other properties of the flower so as to be left with the number of petals. There is nothing more to this abstraction process than a mental activity in conjunction with the reality of the properties in question. For example, claims to the effect that this process of abstraction reveals a transcendent abstract entity or that the properties abstracted have their origins in the human mind itself result from 'mental activities' that have their origins in claims that go beyond the horizon of 'Worlds 1&2', and are of religious or metaphysical nature.

The natural numbers – as a mathematical theory or system – results from a further forma-tive activity of 'World 2' upon the abstraction activity just mentioned. The result is a theoretical system of concepts having the status of a cultural object. Its connection with 'World 1' is such that the consequences of this system are 'realizable' or 'instantiated' in the entities comprising the concrete reality of 'World 1'. For this purpose no abstract entities ontologically distinct from the actual properties of concrete things focussed upon in the activity of mental abstraction are required.

The ways in which the mathematical system of natural numbers (or any other number system for that matter) is instantiated within the concrete 'World 1' requires some comment. That it is not a simple matter of 1+1=2 regardless of context is made evident from the following example:

> *The pair of socks I want to buy costs $2. I tender a $5 note to the shop assistant. How much change should I get?*

The answer in this case is simply an application of the subtraction operation within the number system to get the result: $3. However, what if we include the following conditions to the problem?

- *(i) The economic conditions in which the transaction takes place is akin to that of Germany during the mid 1920's: the inflation rate, measured in annual terms, is of the order of 500,000 %.*
- *(ii) The shop assistant finds that there is no small change in the store, and asks me to come back in the afternoon. (It is currently 9 a.m.).*

Quite clearly I would be expecting a bit more than $3 when I returned for my change! To find out just how much I should receive, it is no longer a simple matter of subtracting 2 from 5. The latter problem cannot be solved simply within the 'pure mathematical domain' of the theory of numbers (whether natural, integers or real). It requires a theory that incorporates an economic/financial understanding of inflation to the way in which the numbers are calculated within the number system. In this respect the level of abstraction of the required needs to take these kinds of properties into consideration.

This issue is perhaps an appropriate one to introduce the second demarcation issue that needs to be addressed–that of the distinguishing feature of mathematical theories.

5 The Distinguishing Features of 'Pure' Mathematical Theories.

A stone building and a wooden building are both buildings. They are both made by human beings as socio-cultural objects. They not only differ in what they are made from, they may also differ in what they are made for: for example, one may be a house, the other a prison. A mathematical theory and a chemical theory are both theories; in this they both differ from houses and sculptures. Whilst the ontological status of the former are clearly *contents of thought*, that of the houses and sculptures remains linked (a) with the material from which they are fashioned, and (b) with their intended socio-cultural purpose – what it is they were made for. In this sense both chemical and mathematical theories focus upon the attempt to analyse and understand some of the properties that stones, wood (why one burns and the other doesn't) or street garbage collection systems may have, and, in the course of solving them, developing theories that begin to develop a life of their own - with the problems emerging from the theoretical problem situations providing the stimulus for the further growth of knowledge.

I want to claim that a mathematical theory differs from a chemical theory in a similar manner to the way in which a stone building differs from a wooden building – whilst they are both theories, they differ in the 'material' from which they are built – in the sense of the kinds of properties of 'World 1' with which they deal and focus upon. For the present purposes it will suffice to attempt to state the kinds of properties with which mathematics is concerned.

In an effort to try to state the unhewn building material of mathematics Ludwig Kronecker made the well-known statement to the effect that "The integers were created by God; all else is the work of man." This statement may be taken as a conjecture to the effect that 'the integers - like gold - are discovered, whilst all else in mathematics – like gold rings, goblets, plates and other vanities – are made from the precious raw material of the integers.

It is precisely this kind of venture that leads to foundationism – the effort to find the basic building blocks, and then use them like pack of cards, to rebuild the grand castle of mathematics from the ground up. It is probably for this kind of reason that Hersh wants to abandon the whole attempt to articulate the demarcation line between mathematical and non-mathematical theories.

Taking the cue both from the full range of properties displayed by the concrete things /events/ processes of our experience, as well as a careful appreciation of what mathematicians actually do and produce I want to suggest that the demarcation problem of distinguishing mathematical theories from non-mathematical theories might be solved along the following lines:

Thesis Six:

1. Concrete problems such as that of the Konigsberg Bridges arise from practical life.

2. These problems very often involve only a few of the properties of the concrete entities concerned.

3. Mathematical (as well as other types of theories) arise in the manner illustrated by Euler and the Konigsberg Bridges.

4. The 'building materials' of 'pure' mathematical theories are characterised by:

   - the properties of ***discrete*** unity in the midst of multiplicit**y** demonstrated in the concrete terms of palings in a fence, petals on a flower, leaves on a tree, atoms in a molecule, houses in a street, bridges across a river, land areas divided by rivers etc.
   - the properties of ***continuity*** demonstrated by the internal connectedness of land areas divided from one another by rivers, by legal boundaries etc.

5 Mathematical theories arise by focussing upon these kinds of properties in the manner of abstraction, definition, generalization illustrated by Euler's paper on the Konigsberg Bridges.

6 A recent preliminary conjectural classification of 'pure' mathematical theories based upon this kind of approach has been proposed by Willem Kuyk, professor of Pure Mathematics at the University of Antwerp, and published under the title C*omplementarity in Mathematics.* A summary of standard "pure" mathematical *disciplines* based upon the two *basic property kinds of discreteness and continuity* exemplified in concrete things/events/processes given in this book, is as follows:

Standard mathematical disciplines

(a) Discrete (Algebra)	(b) Continuous (Topology)
Set Theory Logic	
1. Groups, Rings, Graphs; etc.	Topological Groups, Rings etc.
2. Algebraic Geometry	Analytic Geometry
3. Algebraic Topology (Homological Algebra)	Analytic Topology, Manifolds
4. Algebraic Lie Theory	Lie Theory
5. Algebraic Sheaf Theory	Analytic Sheaf Theory
6. Algebraic Number Theory	Analytic Number Theory
7. Algebraic Manifolds	Analytic (& Differentiable) Manifolds
8. Combinational Theory of Probability	Analytic Theory of Probability (Measure Theory)
9. Summability Theories (Series)	Integral Calculus
10. Difference Equations	Differential Equations
11. Linear Algebra	Linear Analysis (Branch Algebras)
12. Finite Geometries	Classical Geometries
13. Computer Sciences)	
Category Theory	

It will be noted that *Graph Theory* is listed under the heading of 'discrete' (algebra), even though Euler and its other pioneers considered many of its problems to be of a geometrical (and therefore continuous) nature. This raises some interesting issues. Firstly, the basic entities - in the sense of 'abstract concepts' - of Graph Theory are two-fold: nodes (vertices) and arcs. The unity and multiplicity of both these kinds of entities comprise the set of *discrete* entities of the system. The set of arcs connecting the nodes comprise the *continuous* links between the nodes. In the case of the Konigsberg Bridges, the abstraction of the non-geometrical properties from the land areas leaves us with a discrete set of mutually exclusive connected topological regions separated by curves arising from the abstraction of the non-geometrical properties of the rivers separating them. On the other hand the arcs connecting the nodes arise from an abstraction of the non-geometrical properties from the bridges connecting the various land areas. Whereas, from the point of view of Graph Theory, the continuity properties of each of the connected topological regions may be further abstracted to leave 'nodes', the same cannot be said for the arcs – these need to retain their continuity properties if the structure of a graph is to be applicable to this concrete situation. Thus the structure of Graph Theory involves both *discrete* and *continuous* entities (nodes and arcs).

Moreover, *the methods* of Graph Theory – as pioneered by Euler – are typical of discrete mathematics rather than continuous mathematics (no limiting processes are involved, for example). However, much the same can be said for many branches of geometry. In other words, whereas Kuyk's preliminary classification tends to convey the idea that the standard mathematical disciplines should be considered as either discrete or continuous, the example of Graph Theory growing out of Euler's Konigsberg Bridges suggests that this is not so. Hence this case study would suggest that this preliminary classification is in need of a lot more careful scrutiny and detailed analysis if it is to be a helpful guide to the classification of pure mathematical fields in a way that can escape the kind of anti-foundationist critique of Reuben Hersh or the essentialist critique in the Popperian strain of thought.

7 Finally, rather than distinguish between 'Pure' and 'Applied' mathematical problems, it would be better to distinguish between 'Concrete' and 'Abstract' problems. As such the concrete problems may serve as a catalyst for the development of mathematical theories, as in the case of Euler and the Konigsberg Bridges. Equally, how-ever, it may be the case that a solution to a theoretical problem may be 'applied' to the concrete circumstances of a practical problem - as with the case of the Garbage Truck Collection problem. Traditional 'Pure Mathematical' disciplines such as those cited above may be said to be concerned with the properties of discreteness and continuity in ways that do not take other kinds of properties directly into consideration. On the other hand more traditional 'Applied Mathematical' disciplines do take physical, economic and other kinds of properties into consideration in the formulation of their theories. Since the properties of discreteness and continuity as investigated by the traditional 'Pure Mathematical' disciplines listed above do not occur within 'World 1' things/events/processes apart from physical and other properties, this feature accounts for their 'more abstract character'. However, this does not imply that their concepts exist simply within 'World 2' or within some Platonic realm of abstract entities. They relate to the properties of discreteness and continuity of the things/processes/events of 'World 1'. Since this is also true of the more traditional 'Applied Mathematical' disciplines but in ways that incorporate other types of properties, the distinction between 'Concrete' and 'Abstract' problems is one that may be made of all mathematical problems.

6 Order, Theology, Religion and Pure Mathematics

Hersh wishes to reject the Foundationist-Platonic heritage in part due to its origins in the close Pythagorean connection between religion, theology, mysticism and mathematics that lies at the fountainhead of this tradition. The implication is that any philosophy of mathematics capable of meeting the criteria he sets should, in its rejection of Platonism, also reject purported connections between religion and mathematics as a vestige of Pythagoreanism. The questions relating to the ultimate origins of the order investigated by human theory development cannot be dismissed as glibly as that. By way of illustration consider the question that Hersh himself devotes some major attention to:

Is mathematics created or discovered?

Platonists take the view that mathematical entities as well as the results that pertain to them can't be created: they already exist, whether we know them or not. In his radical critique of the whole Platonist-Foundationist tradition, Hersh rightly points out the connection of this 'discovery' view with theology and religion. However, both his diagnosis of the problems involved and his solution to them appear to perpetuate a view of religion that, if he is to effectively escape an accusation of dogmatism, is itself in need of critique. For example, with reference to the statement of Kronecker quoted above concerning the integers being created by God, and all else the work of man, he writes:

> *Since Kronecker was a believer, it's possible that he meant this literally. But when mathematicians quote it nowadays, 'God' is a figure of speech. We interpret it to mean: 'The integers are discovered, all else is invented.'*

He further goes on to say:

> *Such a statement is an avowal of Platonism, at least as regards the integers or the natural numbers.*

In this respect he wishes to alter the statement, replacing 'integers' by 'counting numbers', with the result that 'Counting numbers are discovered. Pure numbers were invented' is deemed an acceptable illustration of the difference between discovery and creation. In more general terms he suggests that the *solutions* to *problems* within a theory are *discovered* but the *theories* themselves are *created*.. This distinction is based upon the view that the solution to a problem within a theory is unique, whereas the same thing cannot be said of the development of a theory. The latter is the result of cultural creativity. However, fact that Hersh is able to make the radical claim of Point 2 of Thesis One, namely:

> *There's no need to look for a hidden meaning or definition of mathematics beyond its social-historic-cultural meaning. Social-historic is all it needs to be. Forget immaterial, inhuman 'reality'.*

raises the question regarding the meaning he attaches to the word 'create'. On the face of it he could easily mean 'creatio ex nihilo' as opposed to 'culturally form'. I would suggest that an important difference between the two arises from religious views concerning the ultimate source of order to 'Worlds 1&2'. He does not repudiate either view. To attribute a *creatio ex nihilo* to human theories is to place human beings themselves in the position of providing such order. On the other hand, to view theories as providing fallible, inventive and culturally innovative accounts of the way the cosmos is ordered can either place the source of that order to an immanent 'Nature' or to a Creator, with one or other always having existed. Moreover, such remarks as those above are sufficient to raise the question: Is someone to be labelled a Platonist simply because (a) they are a believer and (b) because they consider that mathematical theories, albeit humanly formed, should be about properties of 'World 1' things/events/processes?

In what follows I shall endeavour to uncover the basic epistemological and/or ontological issues that have been crucial to the development of the Platonist philosophy of mathematics, and, in this light try to suggest a better and fairer way in which theological and religious issues may be considered to function within mathematics, especially in the philosophy of mathematics.

6.1 Understanding the Epistemological and Ontological Roles of Platonism in Mathematical Theories

The Platonist position in respect of mathematical activity always leading *to discovery* is dependent upon a number of epistemological and ontological assumptions:

1. The universe of 'Worlds 1 &2' is organised with reference to eternal Ideas. These Ideas have a divine quality that, on the one hand, transcends 'Worlds 1&2' and yet, on the other, plays a significant role in ordering both of them.
2. By the appropriate exercise of human reason it is possible to enter directly into this world of eternal Ideas, with the consequence that it is thereby possible to discover the lawful order by which this Universe is governed.
3. As a result of adapting its theology to these Platonic or Neo-Platonic notions, Christendom inherited the idea that, via the exercise of abstract Reason in Platonic form, it was possible to know 'the mind of God' in such a way that we humans could have some direct access to the law by which God ordered the cosmos. This was exemplified supremely by mathematics. Thus Aquinas, who was more of a follower of Aristotle than Plato, was able to write that:

    Since mathematics is intermediate between natural and divine science, it is more certain than either of them.

4. On the one hand, for many, the ontological features of Platonism have been so chipped away over the past four hundred years that they are left with a 'main-stream' philosophy of mathematics that continues many of the aspirations to-ward *apriorism* and 'certainty' that arises from this tradition. On the other hand, for those who continue to hold to the view that mathematical entities have an eternal existence apart from concrete entities/events/processes, the activities of mathematics lead to their being 'discovered'.

6.2 Some Issues Arising with the Rejection of Platonism.

In the process of rejecting this Platonic-Foundationist heritage there are at least three central issues that need to be brought out into the open. This is so because of the kinds of implications mentioned above by Hersh to the effect that religion and theology are human activities that (a) should be kept out of secular affairs and (b) are presumed not to be applicable to the ways in which modern secularists themselves pursue secular affairs.

1. What is the ultimate source of the order to 'Worlds 1 &2'? This issue is fundamentally of a religious nature, and, whilst open to discussion and debate, goes more deeply into human life than 'reason' on its own. In this respect it is unfortunate that we continue to be subject to the kind of debate (political point scoring might be a better description) that goes on between evolutionism and creationism. This way of dealing with the issue has simply muddied the waters between religion and science. The more pertinent issue for the former is naturalism versus theism. Those advocating evolutionism very often do so with an unacknowledged baggage of religious naturalism. It is quite possible to be in a minority opposing the latter without

advocating what has unfortunately become known as 'creationism'. Whilst the ways in which religious beliefs – of whatever colour – operate in science is more complicated than 'scientific realists' would usually want to acknowledge, fruitful discussion between people of different beliefs can proceed effectively, but requires an attitude of self-criticism that is willing to acknowledge the ways in which beliefs concerning the ultimate source of the order to 'Worlds 1&2' affect their theorising.

2. What role does the cultural formation of human beings play within the ordering of 'Worlds 1 &2'? This issue too has what might be described as a 'religious flavour'. Human beings have held vastly different convictions regarding what they can/may or can't/shouldn't achieve via their exercise of cultural power. The two extremes vary from an abdication of any such responsibility because of the believed arbitrary power of the gods and the need to propitiate them, on the one hand, to the unabashed bold confidence that the unaided secularist exercise of such power through politics, science, technology and education, would enable humankind to bring about unlimited cultural and social progress.
3. What can human beings hope to know of the ultimate ordering principles of the universe? Few today would want to try to defend the strong epistemo-logical claim of Platonism to the effect that the simple exercise of abstract reason yielded these ultimate secrets! However, a strong case can be made for the human activity of theorising providing a fallible level of insight into the lawful ordering of the cosmos. In this respect, in radical contrast to the assumptions of Platonism, the point needs to be made that human beings do not have direct access to the laws governing the cosmos. We have a direct access only to the particulars of the regularity of the ways in which things/ events/processes operate within 'World 1'. In this respect our scientific theorising efforts may be said to constitute a fallible attempt to gain insight into the lawful ordering of the cosmos.

6.3 Is Mathematics Created or Discovered?

The two primary activities in respect of the socio-cultural life mathematics are problem- solving and theory development. The solution to a problem – with or without the aid of a theory – is quite easily accepted as a *discovery*. The solution was there before we found it; it just took a lot of finding! The more contentious issue arises with respect to the way in which we should describe theory development. Now, the main reasons why, within a Platonic outlook, developments of this kind are also considered *discoveries* are (i) that the lawful ordering of the cosmos via transcendent abstract mathematical entities may be equated with the human theories that have been developed and (ii) that these abstract entities have always existed. To reject Platonism does not necessarily imply the other extreme – namely that mathematical theories are free constructions having little or nothing to do with the lawful ordering of the cosmos.

Consider a comparison between Euler's Konigsberg Bridges and the use of Newtonian theory in the discovery of Neptune. As mentioned in section 4.3, the planet Neptune was discovered with the help of Newtonian theory. This involved the postulation of a planet with appropriate mass and orbit so that certain anomalies in the orbits of known planets could be accounted for. A telescope was then pointed to look in the appropriate place. Lo and behold, the planet was there! Who discovered it – the theoretician or the astronomer?
The answer has to be that both played a part in its discovery. In this respect, too, the contribution played by Newton and his predecessors should not be overlooked. There is a similarity between this case and that of Euler's discovery to the effect that no continuous path existed crossing each of the Konigsberg bridges once and only once. In both cases a theory played an important role

and in each case an empirical fact was involved. How-ever, there is a significant difference in that the truth of the latter example did not require an empirical observation to back it up. It simply followed as a logical consequence of the theory. Now, while this might bring some joy to the Platonist – a humanly developed theory may, after all, be equated with the lawful ordering of things – we should soberly reckon with the fallibility of all human knowledge. In this respect, whilst the theories of physics, chemistry and other disciplines require much more by way of empirical testing than mathematical theories, the latter too are to be understood as fallible human attempts to understand the lawful ordering of the cosmos, and that this kind of exercise involves the exercise of human formative power by way of the process of abstraction and the innovative formation of theoretical concepts discussed above.

I put forward the conjecture that the issue relating to a less strict requirement for the empirical testing of mathematical theories may have something to do with the fact that the main properties forming the content of its theories – discreteness and continuity – do not occur within 'World 1' without other kinds of properties – physical properties in particular.

To summarise:

Mathematical theories are shaped by human beings as cultural objects (the word 'created' may be used in this sense if required) that fallibly endeavour to uncover ways in which a range of properties (such as those discrete/continuous properties that have become known as 'Graphs') are ordered in the concrete things/events/processes of the cosmos.

Solutions to the mathematical problems – either concrete or abstract – are discovered as the result of employing the appropriate methods (such as the critical application of the Pappusian method of analysis and synthesis).

Hence mathematics is part of human culture and, as such, involves a range of social interactions. The results are in the form of the discovery of solutions to problems and the development of theories that purport to give a fallible account of the lawful ordering of certain properties of the things/events/processes of the cosmos of our experience.

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